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# Approximating partial derivatives of first and second order by quadratic spline quasi-interpolants

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## Abstract

Given a bivariate function  $f$  defined in a rectangular domain  $\Omega$ , we approximate it by a  $C^1$  quadratic spline quasi-interpolant (abbr. QI) and we take partial derivatives of this QI as approximations to those of  $f$ . We give error estimates and asymptotic expansions for these approximations. We also propose a simple algorithm for the determination of stationary points, illustrated by a numerical example.

*Key words:* Quadratic spline quasi-interpolant; Partial derivative approximation; Stationary points detection.

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## 1 Introduction and notations

Let  $\Omega = I \times J = [a, b] \times [c, d]$  be a rectangular domain. For a given steplength  $h > 0$ , we assume that  $b = a + mh$  and  $d = c + nh$  and we consider the uniform partitions  $X_m = \{x_i = a + ih, 0 \leq i \leq m\}$  and  $Y_n = \{y_j = c + jh, 0 \leq j \leq n\}$ , respectively in  $I$  and  $J$ . We also need the midpoints  $s_i = \frac{1}{2}(x_{i-1} + x_i)$  and  $t_j = \frac{1}{2}(y_{j-1} + y_j)$  of subintervals defined by the two partitions. For the sake of simplicity, and in order to avoid the use of boundary B-splines and functionals, we add two points at each end of intervals (see Figure 1), denoted respectively by  $x_{-2}, x_{-1}, x_{m+1}, x_{m+2}$  and  $y_{-2}, y_{-1}, y_{n+1}, y_{n+2}$ . In the same way, we have two

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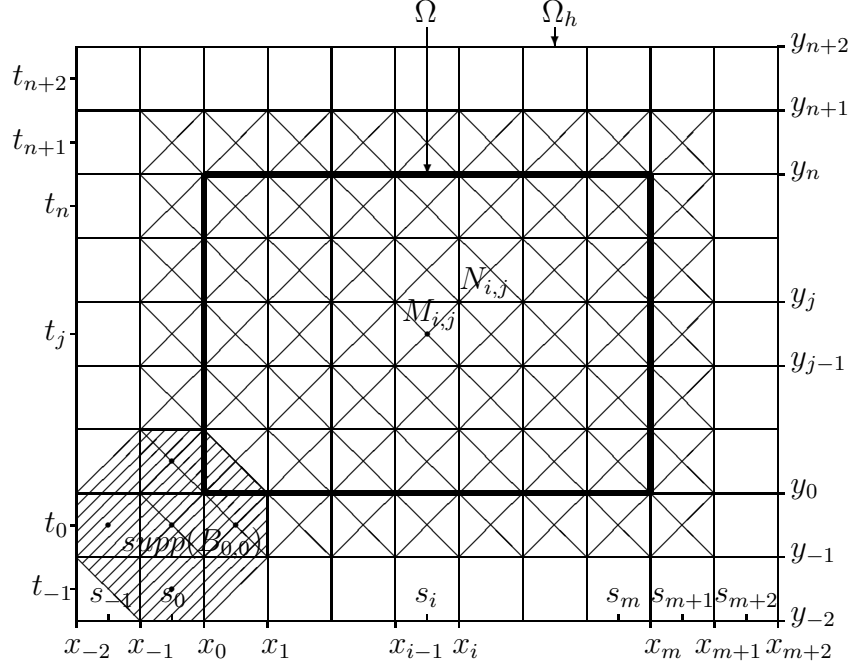


Figure 1. Domains  $\Omega$  and  $\Omega_h$

more midpoints at each end:  $s_{-1}, s_0, s_{m+1}, s_{m+2}$  and  $t_{-1}, t_0, t_{n+1}, t_{n+2}$ . Then we define  $\Omega_h = [a - 2h, b + 2h] \times [c - 2h, d + 2h] = [x_{-2}, x_{m+2}] \times [y_{-2}, y_{n+2}]$ . We assume in the following that  $f$  is defined in the bigger domain  $\Omega_h$ . This domain is endowed with the so-called criss-cross (or type 2) triangulation  $\mathcal{T}_{mn}$  consisting in drawing diagonals in each subsquare  $\Omega_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$  of  $\Omega_h$ . The centres of these subsquares are the points  $M_\alpha = (s_i, t_j)$ , and their vertices are the points  $N_\alpha = (x_i, y_j)$ . Let  $A = \{\alpha = (i, j), 0 \leq i \leq m + 1, 0 \leq j \leq n + 1\}$ ,  $\varepsilon_1 = (1, 0)$  and  $\varepsilon_2 = (0, 1)$ . Then the  $C^1$  quadratic spline quasi-interpolant considered in this paper is defined by

$$Qf = \sum_{\alpha \in A} \mu_\alpha(f) B_\alpha$$

where the coefficient functionals are

$$\mu_\alpha(f) = \frac{3}{2}f(M_\alpha) - \frac{1}{8}(f(M_{\alpha-\varepsilon_1}) + f(M_{\alpha+\varepsilon_1}) + f(M_{\alpha-\varepsilon_2}) + f(M_{\alpha+\varepsilon_2})).$$

and the  $B_\alpha$ 's are the classical  $C^1$ -quadratic B-splines (or box-splines). For their properties, see e.g. [2, 3]. Note that the family  $\{B_\alpha, \alpha \in A\}$  of B-splines whose support intersect  $\Omega$  is only a spanning system, not a basis, of the space  $S_2(\Omega, \mathcal{T}_{mn})$  of quadratic splines defined in  $\Omega$  endowed with the triangulation  $\mathcal{T}_{mn}$ . However, this has no effect on the definitions and results given below. Note also that for the construction of  $Qf$ , we need values of  $f$  at points of

$\{M_\gamma \in \Omega_h, \gamma \in \Gamma\}$  where  $\Gamma = \{\gamma = (i, j), -1 \leq i \leq m+2, -1 \leq j \leq n+2\}$ , except the four pairs  $(-1, -1), (m+2, -1), (-1, n+2), (m+2, n+2)$ .

By introducing the fundamental splines

$$\tilde{B}_\gamma = \frac{3}{2}B_\gamma - \frac{1}{8}(B_{\gamma-\varepsilon_1} + B_{\gamma+\varepsilon_1} + B_{\gamma-\varepsilon_2} + B_{\gamma+\varepsilon_2}),$$

it is also very convenient and useful to write  $Qf$  in the "quasi-Lagrange" form

$$Qf = \sum_{\gamma \in \Gamma} f(M_\gamma) \tilde{B}_\gamma.$$

In particular, the Lebesgue function  $\Lambda$  of  $Q$ , defined by

$$\Lambda = \sum_{\gamma \in \Gamma} |\tilde{B}_\gamma|$$

allows us to evaluate its infinity norm (see Section 3.1 below)

$$\|Q\|_\infty = |\Lambda|_\infty = \max\{\Lambda(x, y), (x, y) \in \Omega\} = 1.5.$$

The operator  $Q$  is constructed in order to be *exact* on the space  $\Pi_2$  of bivariate *quadratic polynomials* (i.e. of total degree at most 2). In other words, it satisfies  $Qp = p$  for all  $p \in \Pi_2$ .

There exist other types of quadratic spline approximants on the same triangulation (see e.g. [9] with applications to contour plotting) and on other types of triangulations, e.g. the Powell-Sabin one ([8]).

Here is an outline of the paper. In Section 2, for the sake of completeness, we give the full equations of fundamental splines  $\tilde{B}_\gamma$  and of their first and second order partial derivatives, in each of the 68 subtriangles of its support. In Section 3, we give error estimates for infinity norms of  $f - Qf$  and its p.d. In Section 4, we give a general algorithm for the exact computation of stationary points of  $Qf$ . Finally, in Section 5, we give a numerical example in order to illustrate this algorithm.

Partial derivatives are denoted by one of the following forms :

$$\begin{aligned} \partial_1 &= \frac{\partial}{\partial x} = D^{10}, & \partial_2 &= \frac{\partial}{\partial y} = D^{01}, \\ \partial_1^2 &= \frac{\partial^2}{\partial x^2} = D^{20}, & \partial_1 \partial_2 &= \frac{\partial^2}{\partial x \partial y} = D^{11}, & \partial_2^2 &= \frac{\partial^2}{\partial y^2} = D^{02}. \end{aligned}$$

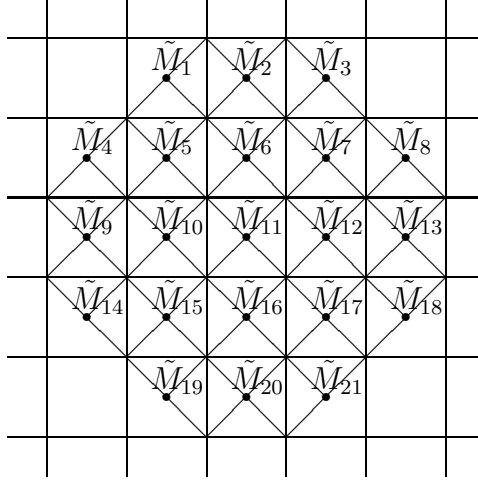


Figure 2. Centres  $\tilde{M}_k$  of the 21 supports of fundamental splines  $\tilde{B}_\gamma$  covering the subsquare  $\Omega_{ij}$  centered at  $M_{ij} = \tilde{M}_{11}$

In Taylor's formulas, we use Ciarlet's notations [5], with  $C_k^j = \frac{k!}{j!(k-j)!}$  :

$$D^k f(M) \cdot (M_0 M)^k = \sum_{j=0}^k C_k^j (x - x_0)^j (y - y_0)^{k-j}.$$

For norms of derivatives, we set, for  $k \geq 1$  :

$$\|D^k f\|_{\infty, \Omega} = \max\{|D^\alpha f|_{\infty, \Omega}, |\alpha| = k\},$$

where , for  $\alpha = (\alpha_1, \alpha_2)$  and  $|\alpha| = \alpha_1 + \alpha_2 = k$  :

$$|D^\alpha f|_{\infty, \Omega} = \max\{|D^\alpha f(M)|, M \in \Omega\}.$$

Often, we use the simpler notation  $|\cdot|_\infty$  instead of  $|\cdot|_{\infty, \Omega}$ , without explicit reference to the domain  $\Omega$ .

## 2 Equations of fundamental splines and of their partial derivatives

The local representation of B-splines and fundamental splines in the Bernstein basis of bivariate quadratic polynomials in each triangle of their support has been given in the technical reports [13b] and [13c]. For the sake of simplicity, we now give the equations of fundamental splines w.r.t. local cartesian coordinates  $(u, v) \in [-\frac{1}{2}, \frac{1}{2}]^2$ , with  $(x, y) = (s_i + uh, t_j + vh)$ , associated with each subsquare  $\Omega_{ij}$  of the partition, the origin being taken at its centre  $M_{ij}$ . We consider only the 21 fundamental splines  $\tilde{B}_\gamma$ , for which the support cover (at least partly)

Table 1  
Equations of  $16\tilde{B}_\gamma$

k	N	S	E	W
1	$-\bar{p}_1^2$	0	0	$-\bar{p}_1^2$
2	$-\frac{1}{2}(1+4\bar{q}_2)$	$-\frac{1}{2}p_3^2$	$p_1^2 - \frac{1}{2}p_3^2$	$\bar{p}_1^2 - \frac{1}{2}p_3^2$
3	$-p_1^2$	0	$-p_1^2$	0
4	$-\bar{p}_1^2$	0	0	$-\bar{p}_1^2$
5	$13\bar{p}_1^2 + 2\bar{p}_1 - 1$	$-(1 - \bar{p}_1)^2$	$-(1 - \bar{p}_1)^2$	$13\bar{p}_1^2 + 2\bar{p}_1 - 1$
6	$4(1 + 6\bar{q}_2)$	$2q_3$	$13\bar{p}_1^2 + 4(1 + 6\bar{q}_2)$	$13p_1^2 + 4(1 + 6\bar{q}_2)$
7	$13p_1^2 - 2p_1 - 1$	$-(1 + p_1)^2$	$13p_1^2 - 2p_1 - 1$	$-(1 + p_1)^2$
8	$-p_1^2$	0	$-p_1^2$	0
9	$\bar{p}_1^2 - \frac{1}{2}\bar{p}_2^2$	$p_1^2 - \frac{1}{2}p_2^2$	$-\frac{1}{2}\bar{p}_2^2$	$-\frac{1}{2}(1 - 4q_1)$
10	$13p_1^2 + 4(1 - 6q_1)$	$13\bar{p}_1^2 + 4(1 - 6q_1)$	$2\bar{q}_4$	$4(1 - 6q_1)$
11	$p_4$	$p_4$	$p_4$	$p_4$
12	$13\bar{p}_1^2 + 4(1 + 6\bar{q}_1)$	$13p_1^2 + 4(1 + 6\bar{q}_1)$	$4(1 + 6\bar{q}_1)$	$2q_4$
13	$p_1^2 - \frac{1}{2}p_2^2$	$\bar{p}_1^2 - \frac{1}{2}p_2^2$	$-\frac{1}{2}(1 + 4\bar{q}_1)$	$-\frac{1}{2}p_2^2$
14	0	$-p_1^2$	0	$-p_1^2$
15	$-(1 - p_1)^2$	$13p_1^2 + 2p_1 - 1$	$-(1 - p_1)^2$	$13p_1^2 + 2p_1 - 1$
16	$2\bar{q}_3$	$4(1 - 6q_2)$	$13p_1^2 + 4(1 - 6q_2)$	$13\bar{p}_1^2 + 4(1 - 6q_2)$
17	$-(1 + \bar{p}_1)^2$	$13\bar{p}_1^2 - 2\bar{p}_1 - 1$	$13\bar{p}_1^2 - 2\bar{p}_1 - 1$	$-(1 + \bar{p}_1)^2$
18	0	$-\bar{p}_1^2$	$-\bar{p}_1^2$	0
19	0	$-p_1^2$	0	$-p_1^2$
20	$-\frac{1}{2}\bar{p}_3^2$	$-\frac{1}{2}(1 - 4q_2)$	$\bar{p}_1^2 - \frac{1}{2}\bar{p}_3^2$	$p_1^2 - \frac{1}{2}bp_3^2$
21	0	$-\bar{p}_1^2$	$-\bar{p}_1^2$	0

the subsquare  $\Omega_{ij}$ . We number them by indices  $k$  from 1 to 21, associated to their support centres  $\tilde{M}_k$ , as given Figure 2 Table 1 gives the equations of fundamental splines  $\tilde{B}_\gamma$ . Tables 2 and 3 give the equations of their first partial derivatives. Finally, Table 4 gives the equations of their second partial derivatives. In Table 1, we use the following abbreviations:

$$\begin{aligned}
p_1 &= u + v, & \bar{p}_1 &= u - v, & p_2 &= 1 + 2u, & \bar{p}_2 &= 1 - 2u, \\
p_3 &= 1 + 2v, & \bar{p}_3 &= 1 - 2v, & p_4 &= 22 - 24(u^2 + v^2), \\
q_1 &= u + v^2, & \bar{q}_1 &= u - v^2, & q_2 &= v + u^2, & \bar{q}_2 &= v - u^2, \\
q_3 &= u^2 + 13v^2 + 12v + 2, & \bar{q}_3 &= u^2 + 13v^2 - 12v + 2,
\end{aligned}$$

Table 2  
Equations of  $16h\partial_1\tilde{B}_\gamma$

k	N	S	E	W
1	$-u + v$	0	0	$-u + v$
2	$2u$	0	$u + v$	$u - v$
3	$-(u + v)$	0	$-(u + v)$	0
4	$-u + v$	0	0	$-u + v$
5	$13(u - v) + 1$	$1 - u + v$	$1 - u + v$	$13(u - v) + 1$
6	$-24u$	$2u$	$-11u - 13v$	$-11u + 13v$
7	$13(u + v) - 1$	$-(1 + u + v)$	$13(u + v) - 1$	$-(1 + u + v)$
8	$-(u + v)$	0	$-(u + v)$	0
9	$1 - u - v$	$1 - u + v$	$1 - 2u$	1
10	$13(u + v) - 12$	$13(u - v) - 12$	$26u - 12$	$-12$
11	$-24u$	$-24u$	$-24u$	$-24u$
12	$13(u - v) + 12$	$13(u + v) + 12$	12	$26u + 12$
13	$-1 - u + v$	$-(1 + u + v)$	-1	$-(1 + 2u)$
14	0	$-(u + v)$	0	$-(u + v)$
15	$1 - u - v$	$13(u + v) + 1$	$1 - u - v$	$13(u + v) + 1$
16	$2u$	$-24u$	$-11u + 13v$	$-11u - 13v$
17	$-1 - u + v$	$13(u - v) - 1$	$13(u - v) - 1$	$-1 - u + v$
18	0	$-u + v$	$-u + v$	0
19	0	$-(u + v)$	0	$-(u + v)$
20	0	$2u$	$u - v$	$u + v$
21	0	$-u + v$	$-u + v$	0

$$q_4 = 13u^2 + v^2 + 12u + 2, \quad \bar{q}_4 = 13u^2 + v^2 - 12u + 2.$$

### 3 Global error estimates for smooth functions

In this section, we assume that  $f \in C^3(\Omega_h)$ : in [13a], error bounds have been already computed in the case of a non-uniform triangulation. In the specific case of uniform triangulations, the constants can be substantially reduced.

Table 3  
Equations of  $16h\partial_2\tilde{B}_\gamma$

k	N	S	E	W
1	$u - v$	0	0	$u - v$
2	-1	$-(1 + 2v)$	$-1 + u - v$	$-(1 + u + v)$
3	$-(u + v)$	0	$-(u + v)$	0
4	$u - v$	0	0	$u - v$
5	$-13(u - v) - 1$	$-1 + u - v$	$-1 + u - v$	$-13(u - v) - 1$
6	12	$26v + 12$	$-13(u - v) + 12$	$12 + 13(u + v)$
7	$13(u + v) - 1$	$-(1 + u + v)$	$13(u + v) - 1$	$-(1 + u + v)$
8	$-(u + v)$	0	$-(u + v)$	0
9	$-u + v$	$u + v$	0	$2v$
10	$13u - 11v$	$-(13u + 11v)$	$2v$	$-24v$
11	$-24v$	$-24v$	$-24v$	$-24v$
12	$-(13u + 11v)$	$13u - 11v$	$-24v$	$2v$
13	$u + v$	$u - v$	$2v$	0
14	0	$-(u + v)$	0	$-(u + v)$
15	$1 - u - v$	$13(u + v) + 1$	$1 - u - v$	$13(u + v) + 1$
16	$26v - 12$	-12	$13(u + v) - 12$	$-13(u - v) - 12$
17	$1 + u - v$	$-13(u - v) + 1$	$-13(u - v) + 1$	$1 + u - v$
18	0	$u - v$	$u - v$	0
19	0	$-(u + v)$	0	$-(u + v)$
20	$1 - 2v$	1	$1 - u - v$	$1 + u - v$
21	0	$u - v$	$u - v$	0

**Theorem 1** *The following error estimates are valid, for constants  $C_0 \leq \frac{5}{96}$ ,  $C_1 \leq 2$  and  $C_2 \leq \frac{41}{4}$  respectively:*

$$\begin{aligned}
|f - Qf|_\infty &\leq C_0 h^3 \|D^3 f\|_\infty, \\
|D^\beta(f - Qf)|_\infty &\leq C_1 h^2 \|D^3 f\|_\infty, \text{ for } |\beta| = 1, \\
|D^\beta(f - Qf)|_{\infty, T} &\leq C_2 h \|D^3 f\|_\infty, \text{ for } |\beta| = 2, \text{ and for all } T \in \mathcal{T}_{mn}.
\end{aligned}$$



Table 4  
Values of second derivatives of  $16h^2\tilde{B}_\gamma$

k	$\partial_1^2$				$\partial_1\partial_2$				$\partial_2^2$			
	N	S	E	W	N	S	E	W	N	S	E	W
1	-1	0	0	-1	1	0	0	1	-1	0	0	-1
2	2	0	1	1	0	0	1	-1	0	-2	-1	-1
3	-1	0	-1	0	-1	0	-1	0	-1	0	-1	0
4	-1	0	0	-1	1	0	0	1	-1	0	0	-1
5	13	-1	-1	13	-13	1	1	-13	13	-1	-1	13
6	-24	2	-11	-11	0	0	-13	13	0	26	13	13
7	13	-1	13	-1	13	-1	13	-1	13	-1	13	-1
8	-1	0	-1	0	-1	0	-1	0	-1	0	-1	0
9	-1	-1	-2	0	-1	1	0	0	1	1	0	2
10	13	13	26	0	13	-13	0	0	-11	-11	2	-24
11	-24	-24	-24	-24	0	0	0	0	-24	-24	-24	-24
12	13	13	0	26	-13	13	0	0	-11	-11	-24	2
13	-1	-1	0	-2	1	-1	0	0	1	1	2	0
14	0	-1	0	-1	0	-1	0	-1	0	-1	0	-1
15	-1	13	-1	13	-1	13	-1	13	-1	13	-1	13
16	2	-24	-11	-11	0	0	13	-13	26	0	13	13
17	-1	13	13	-1	1	-13	-13	1	-1	13	13	-1
18	0	-1	-1	0	0	1	1	0	0	-1	-1	0
19	0	-1	0	-1	0	-1	0	-1	0	-1	0	-1
20	0	2	1	1	0	0	-1	1	-2	0	-1	-1
21	0	-1	-1	0	0	1	1	0	0	-1	-1	0

**Proof** Let us choose a triangle  $T$  of type N(North) and the midpoint  $M_0 = (x_0, y_0)$  of its upper edge. By Taylor's formula, we obtain, for all  $M = (x, y) \in T$ ,

$$f(M) = p_2(M) + r_2(M),$$

where  $p_2$  and  $q_2$  are respectively the quadratic Taylor polynomial and the associated remainder

$$p_2(M) = f(M_0) + Df(M_0).(M_0M) + \frac{1}{2}D^2f(M)(M_0M)^2,$$

$$r_2(M) = \frac{1}{6} \int_0^1 D^3f((1-t)M + tM_0).(M_0M)^3 dt.$$

As  $|M_0M|_1 = |x - x_0| + |y - y_0| \leq h/2$ , we get

$$|r_2(M)| \leq \frac{1}{6} \|D^3f\|_\infty |M_0M|_1^3 \leq \frac{h^3}{48} \|D^3f\|_\infty.$$

On the other hand,  $Qf = Qp_2 + Qr_2 = p_2 + Qr_2$  since  $Q$  is exact on  $\Pi_2$ . Therefore, using the above majoration

$$|f - Qf|_\infty = |r_2 - Qr_2| = |(I - Q)r_2| \leq (1 + \|Q\|_\infty)|r_2| = \frac{5}{2}|r_2| \leq \frac{5h^3}{96} \|D^3f\|_\infty.$$

For  $\partial_1(f - Qf)$ , we can use the following technique. We do not go into details and we only give a sketch of the computations. We start from the expression of the derivative  $\partial_1 Qf = \sum_\gamma f(M_\gamma) \partial_1 \tilde{B}_\gamma$  and from Taylor's formula of order 2:

$$f(M_\gamma) = f(M) + Df(M).(MM_\gamma) + \frac{1}{2}D^2f(M).(MM_\gamma)^2$$

$$+ \frac{1}{6} \int_0^1 D^3f((1-t)M_\gamma + tM).(MM_\gamma)^3 dt.$$

Replacing  $f(M_\gamma)$  by its expansion in the formula of  $\partial_1 Qf$ , we obtain

$$\partial_1 Qf(M) = f(M) + \frac{1}{6} \sum_\gamma \left[ \int_0^1 D^3f((1-t)M_\gamma + tM).(MM_\gamma)^3 dt \right] \partial_1 \tilde{B}_\gamma(M).$$

This is a consequence of the following facts. We use the notation  $M_\gamma = (x_\gamma, y_\gamma)$  and the property of  $Q$  to be exact on  $\Pi_2$ . As  $\sum_\gamma \tilde{B}_\gamma = 1$ ,  $\sum_\gamma x_\gamma \tilde{B}_\gamma = x$ ,  $\sum_\gamma y_\gamma \tilde{B}_\gamma = y$ , then  $\sum_\gamma \partial_1 \tilde{B}_\gamma = 0$ ,  $\sum_\gamma x_\gamma \partial_1 \tilde{B}_\gamma = 1$ , and  $\sum_\gamma y_\gamma \partial_1 \tilde{B}_\gamma = 0$ . Therefore we get

$$\sum_\gamma \partial_1 \tilde{B}_\gamma [Df(M).(MM_\gamma)] = \sum_\gamma \partial_1 \tilde{B}_\gamma [(x_\gamma - x) \partial_1 f(M) + (y_\gamma - y) \partial_2 f(M)] = \partial_1 f(M),$$

In a similar way, as  $\sum_{\gamma} x_{\gamma}^2 \tilde{B}_{\gamma} = x^2$ ,  $\sum_{\gamma} y_{\gamma}^2 \tilde{B}_{\gamma} = y^2$ ,  $\sum_{\gamma} x_{\gamma} y_{\gamma} \tilde{B}_{\gamma} = xy$ , then  $\sum_{\gamma} x_{\gamma}^2 \partial_1 \tilde{B}_{\gamma} = 2x$ ,  $\sum_{\gamma} x_{\gamma} y_{\gamma} \partial_1 \tilde{B}_{\gamma} = y$ ,  $\sum_{\gamma} y_{\gamma}^2 \partial_1 \tilde{B}_{\gamma} = 0$ . and we get

$$\sum_{\gamma} \partial_1 \tilde{B}_{\gamma} D^2 f(M) \cdot [(MM_{\gamma})^2] = 0,$$

We deduce from the above expression that

$$|\partial_1 Qf(M) - f(M)| \leq \frac{1}{6} \|D^3 f\|_{\infty} \sum_{\gamma} |MM_{\gamma}|_1^3 |\partial_1 \tilde{B}_{\gamma}|$$

Now, setting  $\varphi_{\gamma}(M) = |MM_{\gamma}|_1^3 |\partial_1 \tilde{B}_{\gamma}|$ , we use Table 2 to compute the exact maximum of  $\varphi_{\gamma}$  in the triangle  $T$ . We obtain the following bounds:  $\varphi_{\gamma} \leq 9$  for  $k = 1, 3, 4, 8, 9, 13, 15, 17$ ;  $\varphi_{\gamma} \leq 19$  for  $k = 5, 7, 10, 12$ ;  $\varphi_{\gamma} \leq 8$  for  $k = 2, 16$ ;  $\varphi_{\gamma} \leq 12$  for  $k = 6, 11$ , from which we deduce  $\sum_{\gamma} \varphi_{\gamma} \leq 2$  and finally

$$|\partial_1 Qf - f|_{\infty} \leq 2h^2 \|D^3 f\|_{\infty}.$$

A similar method is used for second order derivatives.  $\square$

#### 4 An algorithm for computing stationary points

In view of the results of the preceding sections, we propose the following algorithm for the detection of *stationary points* of  $Qf$ :

- (i) compute  $\pi$  and  $\chi$  at centres  $M_{ij}$  and vertices  $N_{ij}$  of subsquares of the partition.
- (ii) select the subset  $\mathcal{T}' \subset \mathcal{T}_{mn}$  of triangles in which the sum of signs of  $\pi$  at the three vertices is  $\pm 1$ .
- (iii) select the subset  $\mathcal{T}'' \subset \mathcal{T}'$  of those triangles in which the sum of signs of  $\chi$  at the three vertices is  $\pm 1$ .
- (iv) in each triangle of  $\mathcal{T}''$ , solve the system of equations  $\pi = \chi = 0$ , i.e. compute the exact intersection point of the two corresponding segments. We thus obtain the set of stationary points of  $Qf$ .
- (v) study locally the sign of  $H = \sigma^2 - \rho\tau$ : if  $H < 0$ , then if  $\rho > 0$  and  $\tau > 0$ , then there is a local minimum, else if  $\rho < 0$  and  $\tau < 0$ , then there is a local maximum. Finally, if  $H \geq 0$ , we have a saddle point or a degenerate point (see e.g. [7]).

**Remark 2** *Of course, one can exchange the roles of  $\pi$  and  $\chi$  in the above algorithm.*

## 5 Numerical example

We test the previous algorithm with a Scilab ([12]) program and obtain the following results.

**Example 3** *Franke's function* (see e.g. [1], p. 144):

$$f_2(x, y) = .75\exp(-\frac{1}{4}((9x-2)^2 + (9y-2)^2)) + .75\exp(-\frac{1}{49}(9x+1)^2 - \frac{1}{10}(9y+1)) \\ + .5\exp(-\frac{1}{4}((9x-7)^2 + (9y-3)^2)) - .2\exp(-(9x-4)^2 - (9y-7)^2).$$

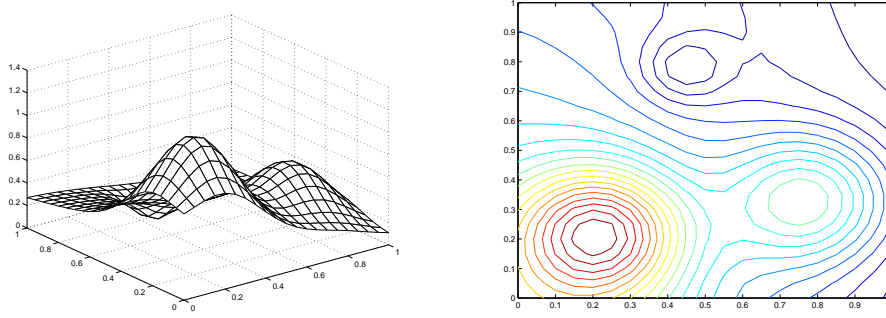


Figure 3. Surface  $z = f(x, y)$  and its level lines

We compute five stationary points  $(x_{i,h}, y_{i,h}, i = 1, \dots, 5$  : one local minimum, two local maxima and two saddle points or degenerate points. For the sake of comparison, we have found with Matlab the following stationary points :  $(x_1, y_1) = (0.456, 0.784)$ ,  $(x_2, y_2) = (0.206, 0.208)$ ,  $(x_3, y_3) = (0.755, 0.326)$ ,  $(x_4, y_4) = (0.556, 0.277)$ ,  $(x_5, y_5) = (0.616, 0.857)$ . Table 5 gives the error  $E_i(h) = ((x_{i,h} - x_i)^2 + (y_{i,h} - y_i)^2)^{1/2}$ ,  $i = 1, \dots, 5$ , for a few values of  $h$ .

Table 5

Approximation errors for the computed stationary points

$h$	$E_1(h)$	$E_2(h)$	$E_3(h)$	$E_4(h)$	$E_5(h)$
1/5	1.03E-01	2.55E-03	4.20E-02	1.22E-02	3.82E-02
1/10	5.38E-03	1.48E-03	2.74E-03	5.39E-03	1.50E-02
1/20	8.26E-04	1.91E-04	2.69E-04	4.32E-04	1.14E-02
1/40	2.52E-04	6.62E-05	7.42E-05	6.53E-05	1.59E-03
1/80	5.51E-05	4.86E-05	1.46E-05	4.74E-05	2.97E-04

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